JOURNAL OF APPROXIMATION THEORY **80**, 76–107 (1995)

Uniform Estimates of Monotone and Convex Approximation of Smooth Functions

KIRILL A. KOPOTUN

Department of Mathematics, University of Alberta, Edmonton, Alberta, Canada T6G 2G1

Communicated by D. Leviatan

Received August 14, 1992; accepted in revised form October 27, 1993

We obtain uniform estimates for monotone and convex approximation of functions by algebraic polynomials in terms of the weighted Ditzian–Totik moduli of smoothness

$$\omega_\varphi^k(f^{(r)}, n^{-1})_{\varphi^r,\infty} := \sup_{0<h\leqslant n^{-1}} \left\| \left(1-\frac{k}{2}h\varphi(x)-x\right)^{r/2} \left(1-\frac{k}{2}h\varphi(x)+x\right)^{r/2} \times \Delta^k_{h\varphi(x)}(f^{(r)}, x)\right\|_{C[-1,1]}$$

where $\varphi(x) := \sqrt{1-x^2}$, for $r \geqslant 3$ and $r \geqslant 5$ in monotone and convex cases, respectively. Together with known results in the positive and negative directions for the other r this complements the investigation of the rate of shape preserving approximation in terms of $\omega_\varphi^k(f^{(r)}, n^{-1})_{\varphi^r,\infty}$ in the sense of the orders of these moduli. It is also shown that some extra conditions on the smoothness of f allow direct results in the cases for which the general estimate in terms of $\omega_\varphi^k(f^{(r)}, n^{-1})_{\varphi^r,\infty}$ is not correct.

1. INTRODUCTION AND MAIN RESULTS

Let Δ^q denote the set of all continuous functions f on $[-1, 1]$ such that $\bar{\Delta}_h^q(f, x) \geqslant 0$ for given $q \in \mathbb{N}$, for all $0 \leqslant h \leqslant 2q^{-1}$, and $x \in [-1, 1-qh]$, where $\bar{\Delta}_h^q(f, x) := \sum_{i=0}^q (-1)^{q-i} \binom{q}{i} f(x+ih)$ is the usual qth forward difference. Then Δ^1 and Δ^2 are the sets of all monotone and convex functions, respectively.

Shape preserving approximation is the approximation of functions $f \in \Delta^q$ by polynomials with nonnegative qth derivatives.

0021-9045/95 $6.00

The present paper is devoted to the investigation of monotone and convex approximation, i.e., cases for $q=1$ and $q=2$.

The rates of the best nth degree unconstrained and shape preserving polynomial approximation of a function f are defined by

$$E_n(f)=\inf_{p_n\in P_n}\|f-p_n\|_\infty,$$

$$E_n^{(q)}(f)=\inf_{p_n\in P_n\cap\Delta^q}\|f-p_n\|_\infty,\qquad q\in N,$$

respectively, where P_n is the set of algebraic polynomials of degree n.

We recall that

$$\omega_k(f,t,[a,b]):=\sup_{0<h\leqslant t}\max_{[x,x+kh]\subset[a,b]}|\bar{\Delta}_h^k(f,x)|$$

denotes the usual kth modulus of smoothness of f.

The Ditzian–Totik modulus of smoothness is given by

$$\omega_\varphi^k(f,t)_p=\sup_{0<h\leqslant t}\|\Delta^k_{h\varphi(x)}(f,x)\|_p,\qquad \varphi(x):=\sqrt{1-x^2}.$$

The Ditzian–Totik weighted modulus of smoothness with weight φ^r is

$$\begin{aligned}\omega_\varphi^k&(f,t)_{\varphi^r,p}\\ &:=\sup_{0<h\leqslant t}\left\|\left(1-\frac{k}{2}h\varphi(x)-x\right)^{r/2}\left(1-\frac{k}{2}h\varphi(x)+x\right)^{r/2}\right.\\ &\qquad\left.\times\Delta^k_{h\varphi(x)}(f,x)\right\|_p,\end{aligned}$$

where

$$\Delta_h^k(f,x):=\begin{cases}\sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i}f(x+(i-k/2)h) & \text{if } |x\pm kh/2|\leqslant 1\\ 0 & \text{otherwise}\end{cases}$$

is the symmetric kth difference.

Let $I:=[-1,1]$; $\rho(h,y):=h\sqrt{1-y^2}+h^2$, $y\in I$, $h\geqslant 0$; $\rho:=\rho(h,x)$, $x\in I$.

For the sake of brevity and convenience of exposition in the uniform metric we shall use the following definition of the "nonuniform" modulus of smoothness $\bar{\omega}_\varphi^k(f,t)$ and the "nonuniform" weighted modulus $\bar{\omega}^k_{\varphi,r}(f,t)$, which are equivalent to $\omega_\varphi^k(f, t)_\infty$ and $\omega_\varphi^k(f, t)_{\varphi^r, \infty}$, respectively (see [5, 14]):

$$\bar{\omega}_\varphi^k(f, t, [a, b]) := \sup_{0<h\leqslant t} \max_{[x, x+k\rho]\subset[a, b]} |\bar{\Delta}_\rho^k(f, x)|, \qquad t \geqslant 0,$$

$$\bar{\omega}_\varphi^k(f, t) := \bar{\omega}_\varphi^k(f, t, I),$$

$$\bar{\omega}_{\varphi, r}^k(f, t) := \sup_{0<h\leqslant t} \sup_{(x, x+k\rho)\subset I} |w_r(x, k, h)\, \bar{\Delta}_\rho^k(f, x)|,$$

where $w_r(x, k, h) := (1+x)^{r/2}(1-x-k\rho)^{r/2}$, $(r+1)\in N$. Obviously, $\bar{\omega}_{\varphi,0}^k(f, t) = \bar{\omega}_\varphi^k(f, t)$.

For $k=0$ let $\bar{\omega}_{\varphi, r}^0(f, t) := \operatorname{ess\,sup}_{x\in(-1,1)} |(1-x^2)^{r/2} f(x)|$.

Throughout the paper C_i, C denote positive constants which are independent of f and n. In order to emphasize that the constant C depends only on $\mu_1, ..., \mu_m$, the expression $C = C(\mu_1, ..., \mu_m)$ will be used.

All constants C are not necessarily the same even when they occur on the same line, but C_i constants are fixed and denote definite quantities throughout the paper.

For arbitrary $f\in C(-1, 1)$, the function $\bar{\omega}_{\varphi, r}^k(f, t)$ can be unbounded. However, it was shown in [5] (see also [14]) that the necessary and sufficient condition for $\bar{\omega}_{\varphi, r}^k(f, t)$ to be bounded for all $t>0$ is the existence of a constant $M<\infty$ such that

$$|(1-x^2)^{r/2} f(x)| < M, \qquad x\in(-1, 1).$$

Let B^r, $r+1\in N$, denote the space of all functions f such that $f\in C[-1, 1]\cap C^r(-1, 1)$ and $|(1-x^2)^{r/2} f^{(r)}(x)| < \infty$, $x\in(-1, 1)$. Thus

$$\bar{\omega}_{\varphi, r}^k(f^{(r)}, t) < \infty, \qquad t>0 \Leftrightarrow f\in B^r.$$

In order to avoid considerations of trivial cases (when the right-hand sides of estimates are equal to infinity) we shall have the restriction on f that it be from the B^r class. Also, let us note that for such functions f, any $l=\overline{0, r}$, and $k\geqslant 0$, the following inequality holds (see also Lemma B below):

$$\bar{\omega}_{\varphi, l}^{k+r-l}(f^{(l)}, t) \leqslant C_0(r, k)\, t^{r-l} \bar{\omega}_{\varphi, r}^k(f^{(r)}, t), \qquad t>0. \tag{1}$$

For unconstrained approximation the following direct result is known.

THEOREM A (see, for example, [5] and [14]). *Let $k\in N$, $(r+1)\in N$. Then for a given function $f\in B^r$ on I and each $n\geqslant k+r-1$,*

$$E_n(f) \leqslant C n^{-r} \bar{\omega}_{\varphi, r}^k(f^{(r)}, n^{-1}), \qquad C = C(r, k). \tag{2}$$

Our goal is to investigate the possibility of obtaining the estimate (2) for shape preserving approximation.

First of all, the following negative results are known.

LEMMA 1. *There is no such constant C that for every nondecreasing function f on I, $f \in B^2$, the estimate $E_n^{(1)}(f) \leqslant Cn^{-2}\bar{\omega}_{\varphi,2}^{1}(f'', n^{-1})$ is valid.*

Moreover, even the estimate $E_n^{(1)}(f) \leqslant C\bar{\omega}_{\varphi,2}^{1}(f'', 1)$ is false.

Thus the estimates $E_n^{(1)}(f) \leqslant Cn^{-l}\bar{\omega}_{\varphi,l}^{k+2-l}(f^{(l)}, n^{-1})$, $C = C(k)$, generally speaking, are not correct for $0 \leqslant l \leqslant 2$ and $k \in N$.

LEMMA 2. *Let $v \geqslant 0$ be fixed. There is no such constant C that for every convex function f on I, $f \in B^4$, the estimate $E_n^{(2)}(f) \leqslant Cn^{-4}\bar{\omega}_{\varphi,4}^{v}(f^{(4)}, n^{-1})$ is valid.*

Moreover, even the estimate $E_n^{(2)}(f) \leqslant C\bar{\omega}_{\varphi,4}^{v}(f^{(4)}, 1)$ is false.

Thus the estimates $E_n^{(2)}(f) \leqslant Cn^{-l}\bar{\omega}_{\varphi,l}^{k+3-l}(f^{(l)}, n^{-1})$, $C = C(k)$, generally speaking, are not correct for $0 \leqslant l \leqslant 4$ and $k \in N$.

Proof. Lemma 1 follows from [10, Lemma 2] and the estimate $\bar{\omega}_{\varphi,2}^{1}(g_b'', t) \leqslant 4$ from the proof of Lemma 3 in [10]. Lemma 2 is a consequence of [8, Theorem 2].

It is worth mentioning that for the particular cases $l = 0$ and $l = 0$ or 1 in Lemmas 1 and 2, respectively, the lemmas follow from A. S. Shevdov's work [15]. ∎

It will be shown in the present paper that the estimate (2) can be obtained for shape preserving approximation of functions $f \in B^r$ with $r \geqslant 3$ and $r \geqslant 5$ in the monotone and convex cases, respectively. For the other r such direct results are known (see [7–9, 11, 12]).

Namely, the following theorems will be proved.

THEOREM 1. *Let $k \in N$, $r \in N$, $r \geqslant 3$, and $f \in B^r$. If a function f is nondecreasing on I, then for every $n = r + k - 1$, $r + k$, ...*

$$E_n^{(1)}(f) \leqslant Cn^{-r}\bar{\omega}_{\varphi,r}^{k}(f^{(r)}, n^{-1}), \qquad C = C(r, k).$$

THEOREM 2. *Let $k \in N$, $r \in N$, $r \geqslant 5$, and $f \in B^r$. If a function f is convex on I, then for every $n = r + k - 1$, $r + k$, ...*

$$E_n^{(2)}(f) \leqslant Cn^{-r}\bar{\omega}_{\varphi,r}^{k}(f^{(r)}, n^{-1}), \qquad C = C(r, k).$$

Now one can summarize estimates of monotone and convex approximation in terms of $\bar{\omega}_{\varphi,r}^{k}$. For the sake of convenience we shall present the results obtained in the form of Figs. 1 and 2. A cross in the position (k, r) means that for a monotone $(i = 1)$ or convex $(i = 2)$ function f from the B^r class the estimate $E_n^{(i)}(f) \leqslant Cn^{-r}\bar{\omega}_{\varphi,r}^{k}(f^{(r)}, n^{-1})$, $C = C(r, k)$, holds. A circle means that this estimate is correct for not all $f \in B^r$.

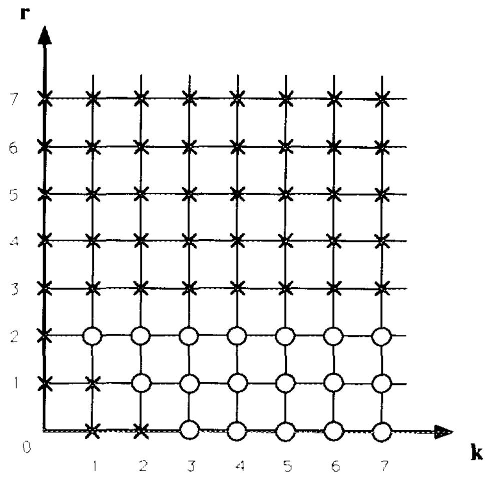

FIG. 1. Monotone approximation.

These results are obtained or are derived from the following papers.

Positive results (monotone case)

- $r=0$, $k=2$, and, consequently, also for $\{(k,r) \mid 1 \leqslant k+r \leqslant 2\}$ (Leviatan [11])
- $r \geqslant 3$, $k=0$ (Dzubenko *et al.* [7])
- $r \geqslant 3$, $k \geqslant 1$ (present paper)

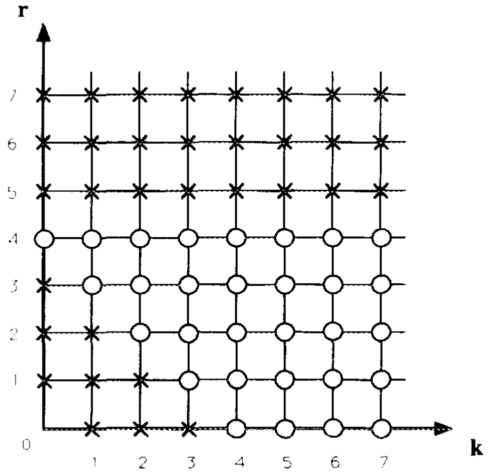

FIG. 2. Convex approximation.

Negative results (monotone case)

$r=0$, $k\geqslant 3$ (Shedov [15])

$r=2$, $k\geqslant 1$ and $r=1$, $k\geqslant 2$ (Kopotun and Listopad [10])

Positive results (convex case)

$r=0$, $k=2$, *and, consequently, also for* $\{(k,r) \mid 1\leqslant k+r\leqslant 2\}$ (Leviatan [12])

$r=0$, $k=3$, and, consequently, also for $\{(k,r) \mid k+r=3\}$ (Kopotun [9])

$r\geqslant 5$, $k=0$ (Kopotun [8])

$r\geqslant 5$, $k\geqslant 1$ (present paper)

Negative results (convex case)

$r=0$, $k\geqslant 4$ and $r=1$, $k\geqslant 3$ (Shvedov [15])

$r=4$, $k=0$, and, consequently (see (1)), also for $\{(k,r) \mid k+r\geqslant 4, r\leqslant 4\}$ (Kopotun [8], see also Lemma 2)

Thus, investigation of the rate of shape preserving approximation of functions from B^r classes in terms of $n^{-r}\bar{\omega}^k_{\varphi,r}(f^{(r)}, n^{-1})$ is complete in the sense of the orders of moduli of smoothness.

However, more detailed consideration shows that some extra conditions on the smoothness of f sometimes allow direct results in the cases for which the general estimate is not correct. These conditions are given by the relation of f to $B^r\bar{H}[k,\psi]$ classes. The necessary definitions and detailed discussions are given in the following section.

2. Shape Preserving Approximation of Functions from $B^r\bar{H}[k,\psi]$ Classes

The following construction of Φ^k classes which was created by Stechkin (see [16], for example) will be useful.

Let Φ^k be the class of all k majorant functions, i.e., continuous non-decreasing functions $\psi=\psi(t)$ on $[0,\infty)$ such that $\psi(0)=0$ and $t^{-k}\psi(t)$ does not increase on $[0,\infty)$.

Obviously, $\bar{\omega}^k_{\varphi,r}(f,t)$ does not have to belong to the Φ^k class. However, the following result is valid: For the function

$$\omega^*(t) := \sup_{u>t} \frac{t^k \bar{\omega}^k_{\varphi,r}(f,u)}{u^k}, \qquad t\geqslant 0,$$

the inequalities

$$\bar{\omega}^k_{\varphi,r}(f,t) \leqslant \omega^*(t) \leqslant C(k)\,\bar{\omega}^k_{\varphi,r}(f,t)$$

hold, and if $\bar{\omega}^k_{\varphi,r}(f,t) \to 0$ as $t \to 0$, then $\omega^* \in \Phi^k$.

Also, for any $\psi \in \Phi^k$ or $\psi \sim 1$, $k \in N$, and $r+1 \in N$ there exists a function $f \in C(-1,1)$ such that

$$C(k)\,\psi(t) \leqslant \bar{\omega}^k_{\varphi,r}(f,t) \leqslant C(k)\,\psi(t).$$

(Proofs of these statements can be found, for example, in [14].)

Now let $B^r\bar{H}[k,\psi]$ be the set of functions $f \in B^r$ such that

$$\bar{\omega}^k_{\varphi,r}(f^{(r)},t) \leqslant \psi(t), \qquad \text{where} \quad \psi \in \Phi^k \text{ or } \psi \sim 1.$$

Thus we have the decomposition of the B^r class

$$B^r = \bigcup_{\psi \in \{\psi \mid \psi \in \Phi^k \text{ or } \psi \sim 1\}} B^r\bar{H}[k,\psi],$$

which, first of all, is complete, as for any function $f \in B^r$ there exists $\psi \in \Phi^k$ or $\psi \equiv \text{const}$ such that $f \in B^r\bar{H}[k,\psi]$ and $\bar{\omega}^k_{\varphi,r}(f^{(r)},t) \sim \psi(t)$, $t>0$. Second, this decomposition is "relatively precise," as for any $\psi \in \Phi^k$ or $\psi \sim 1$ there exists a function $f \in B^r$ such that $\psi(t) \sim \bar{\omega}^k_{\varphi,r}(f^{(r)},t)$, $t>0$.

Taking into account all this and also the inequality (1), one can conclude that Theorems 1 and 2 are corollaries of the following Theorems 3 and 4.

THEOREM 3. *Let $k \in N$, $\psi \in \Phi^k$ or $\psi \equiv 1$, and let $f \in B^3\bar{H}[k,\psi]$ be a nondecreasing function on $[-1,1]$. Then for every $n = k+2, k+3, \ldots$*

$$E_n^{(1)}(f) \leqslant Cn^{-3}\psi(n^{-1}), \qquad C = C(k).$$

THEOREM 4. *Let $k \in N$, $\psi \in \Phi^k$ or $\psi \equiv 1$, and let $f \in B^5\bar{H}[k,\psi]$ be a convex function on $[-1,1]$. Then for every $n = k+4, k+5, \ldots$*

$$E_n^{(2)}(f) \leqslant Cn^{-5}\psi(n^{-1}), \qquad C = C(k).$$

Remark. For $\psi \equiv 1$, Theorems 3 and 4 are consequences of the results obtained in [7] and [8], respectively. In this paper only the case $\psi \in \Phi^k$ will be considered.

Also, as was shown in [10] (see also [8]), functions f which are being discussed in Lemmas 1 and 2 belong to the classes $B^2\bar{H}[1,\text{const}]$ and $B^4\bar{H}[1,\text{const}] \cap B^3\bar{H}[1,Ct]$, respectively.

Now using the following inclusions which are consequences of (1),

$$B^2\bar{H}[1,1] \subset B^l\bar{H}[3-l, Ct^{2-l}], \qquad l=0 \quad \text{or} \quad 1,$$

and

$$B^3\bar{H}[1, t] \subset B^l\bar{H}[4-l, Ct^{4-l}], \qquad 0 \leqslant l \leqslant 2,$$

and also the fact that

$$\psi \in \Phi^k \Rightarrow \psi(t) \geqslant Ct^k, \qquad 0 < t \leqslant 1, \qquad C = \text{const},$$

one can obtain the following lemmas.

LEMMA 3. *There is no such constant C that the estimate $E_n^{(1)}(f) \leqslant Cn^{-2}\psi(n^{-1})$ is valid for every nondecreasing function from the $B^2\bar{H}[1, \psi]$ class with $\psi(t) \sim 1$, $0 < t \leqslant 1$. Thus for $l = \overline{0, 2}$ and any $k \geqslant 3 - l$ the estimate $E_n^{(1)}(f) \leqslant Cn^{-l}\psi(n^{-1})$, $C = C(k)$, generally speaking, is not correct for nondecreasing functions from $B^l\bar{H}[k, C\psi]$ with $\psi(t) \geqslant t^{2-l}$, $0 < t \leqslant 1$.*

LEMMA 4. *There is no such constant C that the estimate $E_n^{(2)}(f) \leqslant Cn^{-3}\psi(n^{-1})$ is valid for every convex function from the $B^3\bar{H}[1, \psi]$ class with arbitrary $\psi \in \Phi^1$. Thus for $l = \overline{0, 3}$ and any $k \geqslant 4 - l$ the estimate $E_n^{(2)}(f) \leqslant Cn^{-l}\psi(n^{-1})$, $C = C(k)$, generally speaking, is not correct for convex functions from $B^l\bar{H}[k, \psi]$ with arbitrary $\psi \in \Phi^k$.*

For $l = 4$ and any $k \geqslant 1$ the estimate $E_n^{(2)}(f) \leqslant Cn^{-4}\psi(n^{-1})$ is not correct for convex functions from $B^4\bar{H}[1, \psi]$ with $\psi(t) \sim 1$, $0 < t \leqslant 1$.

At the same time, the following theorems are valid.

THEOREM 5. *Let $k \in N$ and let f be a nondecreasing function such that $f \in B^2\bar{H}[k, \psi]$ where $\psi(t) = t^\beta$, $t > 0$, $0 < \beta \leqslant k$, and $\beta < 2$. Then for every $n = k+1, k+2, \ldots$*

$$E_n^{(1)}(f) \leqslant Cn^{-2}\psi(n^{-1}) \qquad (\textit{i.e.}, E_n^{(1)}(f) \leqslant Cn^{-2-\beta}),$$

where $C = C(k)(1/(2-\beta))$.

THEOREM 6. *Let $k \in N$ and let f be a convex function such that $f \in B^4\bar{H}[k, \psi]$, where $\psi(t) = t^\beta$, $t > 0$, $0 < \beta \leqslant k$, and $\beta < 4$. Then for every $n = k+3, k+4, \ldots$*

$$E_n^{(2)}(f) \leqslant Cn^{-4}\psi(n^{-1}) \qquad (\textit{i.e.}, E_n^{(2)}(f) \leqslant Cn^{-4-\beta}),$$

where $C = C(k)(1/(4-\beta))$.

We do not know whether Theorems 5 and 6 are valid for all k majorant functions ψ from Φ^k (or even for $\psi(t) = t^\beta$ with other β).

At the same time, it follows from Lemma 4 that in the case for convex approximation for $B^r\bar{H}[k, \psi]$ classes with $0 \leqslant r \leqslant 3$ the negative results are complete since direct theorems are not valid for any k majorant function ψ. The monotone case is still open for research. For example, little or even nothing seems to be known about monotone approximation of functions from the $B^1\bar{H}[2, t^2]$ class.

It is also worth mentioning that Theorems 3–6 are generalizations of the direct results for

$$\hat{H}^\alpha := \begin{cases} B^r\bar{H}[1, t^\beta] & \text{if } \alpha \notin N, \text{ where } r := [\alpha] \quad \text{and} \quad \beta := \alpha - r \\ B^r\bar{H}[2, t] & \text{if } \alpha \in N, \text{ where } r := \alpha - 1 \end{cases}$$

classes which are obtained in [10].

3. Characterization of $B^r\bar{H}[k, \psi]$ Classes

We shall write $\psi \in S(r, k)$ ([1, Conditions Z and Z_k]; see also [16]) if

$$r \int_0^t \psi(u)\, u^{-1}\, du + t^k \int_t^1 \psi(u)\, u^{-k-1}\, du = O(\psi(t)), \qquad t \in (0, 1].$$

The following inverse theorem is known (for example, see [5] and [14]).

Theorem B. *Let $k \in N$, $(r+1) \in N$, and $\psi \in \Phi^k \cap S(r, k)$. If for a given function f on $[-1, 1]$ and each $n \geqslant k + r - 1$ the inequality $E_n(f) \leqslant n^{-r}\psi(n^{-1})$ holds (if $n = 0$ then we define the right-hand side of this inequality to be an absolute constant), then*

$$f \in B^r\bar{H}[k, C\psi], \qquad C = C(r, k).$$

Now a consequence of Theorem B and the direct results for the shape preserving approximation which are given above is the following constructive characteristic of $B^r\bar{H}[k, \psi]$ classes with $\psi \in \Phi^k \cap S(r, k)$.

Theorem 7. *Let $\psi \in \Phi^k \cap S(r, k)$ where*

$$(k, r) \in \{(k, r) \mid k \in N, r \geqslant 3\} \cup \{(k, r) \mid k + r \leqslant 2, k \in N, r + 1 \in N\}.$$

A function f is nondecreasing and in the $B^r\bar{H}[k, C\psi]$ class, if and only if for each $n = k + r - 1, k + r, \ldots$

$$E_n^{(1)}(f) \leqslant Cn^{-r}\psi(n^{-1}), \qquad \text{where} \quad C = C(r, k).$$

THEOREM 8. *Let* $\psi \in \Phi^k \cap S(r, k)$ *where*

$$(k, r) \in \{(k, r) \mid k \in N, r \geqslant 5\} \cup \{(k, r) \mid k + r \leqslant 3, k \in N, r+1 \in N\}.$$

A function f is convex and in the $B^r \bar{H}[k, C\psi]$ *class, if and only if for each* $n = k + r - 1, k + r, \ldots$

$$E_n^{(2)}(f) \leqslant Cn^{-r}\psi(n^{-1}), \qquad where \quad C = C(r, k).$$

Remark. For $(k, r) \in \{(k, r) \mid k + r \geqslant 4, 0 \leqslant r \leqslant 3\}$ and any $\psi \in \Phi^k \cap S(r, k)$ Theorem 8 is false.

For $(k, r) \in \{(k, 0) \mid k \geqslant 3\} \cup \{(k, 1) \mid k \geqslant 2\}$ and $\psi \in \Phi^k \cap S(r, k)$ such that $\psi(t) \geqslant Ct^{k-1}$, $t > 0$, Theorem 7 is false.

For $r = 2$ Theorem 7 is valid in the case $\psi(t) = t^\beta \in \Phi^k$ and $\beta < 2$, and for $r = 4$ Theorem 8 is valid in the case $\psi(t) = t^\beta \in \Phi^k$ and $\beta < 4$, with the same dependance of the constants C on k and β as in Theorems 5 and 6, respectively. In the other cases this question is still open.

4. AUXILIARY NOTATIONS AND DEFINITIONS

Throughout the paper the following notations and definitions will be used (cf. [7–10, 13, 14]):

$$\Delta := \rho(n^{-1}, x), \qquad x \in I;$$

$$x_j := \cos\frac{j\pi}{n}, \quad j = \overline{0, n}; \qquad \bar{x}_j := \cos\left(\frac{j\pi}{n} - \frac{\pi}{2n}\right), \quad j = \overline{1, n};$$

$$x_j^\circ := \cos\left(\frac{j\pi}{n} - \frac{\pi}{4n}\right), \qquad \text{if } j < \frac{n}{2}, \qquad x_j^\circ := \cos\left(\frac{j\pi}{n} - \frac{3\pi}{4n}\right), \qquad \text{if } j \geqslant \frac{n}{2};$$

$$I_j := [x_j, x_{j-1}], \qquad j = \overline{1, n};$$

$t_{j,n} := (x - x_j^\circ)^{-2} \cos^2(2n \arccos x) + (x - \bar{x}_j)^{-2} \sin^2(2n \arccos x)$ is the algebraic polynomial of degree $4n - 2$.

$$T_{j,n}(x) := \int_{-1}^{x} t_{j,n}^{3\chi}(y)\, dy \left(\int_{-1}^{1} t_{j,n}^{3\chi}(y)\, dy\right)^{-1},$$

$$\tilde{T}_{j,n}(x) := \int_{-1}^{x} (y - x_j)(x_{j-1} - y)\, t_{j,n}^{3\chi+1}(y)\, dy$$

$$\times \left(\int_{-1}^{1} (y - x_j)(x_{j-1} - y)\, t_{j,n}^{3\chi+1}(y)\, dy\right)^{-1}$$

are algebraic polynomials of degree $6\chi(2n-1)+1$ and $6\chi(2n-1)+4n+1$, respectively.

$$\sigma_{j,n}(x) := \int_{-1}^{x} (\alpha_1 T_{j,n}(y) + (1-\alpha_1)\, T_{j+1,n}(y))\, dy$$

and

$$\tilde{\sigma}_{j,n}(x) := \int_{-1}^{x} (\alpha_2 \tilde{T}_{j,n}(y) + (1-\alpha_2)\, \tilde{T}_{j+1,n}(y))\, dy, \qquad j=\overline{1, n-1},$$

where numbers α_1 and α_2, $0 \leqslant \alpha_1 \leqslant 1$, $0 \leqslant \alpha_2 \leqslant 1$, are chosen so that $\sigma_{j,n}(1) = \tilde{\sigma}_{j,n}(1) = 1 - x_j$ (see [8]), are polynomials of degree $6\chi(2n-1)+2$ and $6\chi(2n-1)+4n+2$, respectively.

$$J_{n,\xi}(t) = \left(\frac{\sin nt/2}{\sin t/2}\right)^{2\xi+2} \left(\int_{-\pi}^{\pi} \left(\frac{\sin nt/2}{\sin t/2}\right)^{2\xi+2} dt\right)^{-1}$$

is the Jackson type kernel.

$$D_{\zeta,n,\xi}(y,x) = \frac{1}{(\zeta-1)!} \frac{\partial^\zeta}{\partial x^\zeta} (x-y)^{\zeta-1} \int_{\arccos x - \arccos y}^{\arccos x + \arccos y} J_{n,\xi}(t)\, dt, \qquad x, y \in I,$$

is the Dzjadyk type kernel.

χ, ξ, and ζ in the definitions above are integers which will be chosen later.

$L_n(x, f; t_1, t_2, ..., t_{n+1})$ denotes the Langrangean polynomial, of degree not exceeding n, interpolating the function $f(x)$ at the points $t_1, t_2, ..., t_{n+1}$.

For brevity we denote

$$L_k(x, f, \{x_0, h\}) \\ := L_k(x, f, x_0, x_0 + \rho(x_0, h), ..., x_0 + k\rho(x_0, h)).$$

Also, for $a \neq b$ and $(x-a)(x-b) \leqslant 0$ let

$$S(x, l; a, b) := \int_a^x (y-a)^l (b-y)^l\, dy \left(\int_a^b (y-a)^l (b-y)^l\, dy\right)^{-1},$$

$S(x, l; a, b) = 0$ if $(x-a)(x-b) > 0$ and $|x-a| < |x-b|$, and $S(x, l; a, b) = 1$ otherwise.

Without further mention the following inequalities will be used:

$$(x_j - x_{j+1})/3 < x_{j-1} - x_j < 3(x_j - x_{j+1}), \qquad j = \overline{1, n-1};$$
$$\Delta < x_{j-1} - x_j < 5\Delta \qquad \text{for} \quad x \in I_j.$$

5. Auxiliary Statements

In our proofs we shall use the method from [13] (see also [7, 8]) which is a modification of DeVore's ideas concerning the decomposition of the approximated function (see [3, 4]).

The following analog of Whitney's theorem in terms of "nonuniform" moduli of smoothness $\bar{\omega}_\varphi^k$ will be important for the proofs given below.

Lemma A (see [14, Lemma 18.2 and (18.13)]). *Denote* $\rho_0 := \rho(h, x_0)$. *Let* $[x_0, x_0 + (k-1)\rho_0] \subset [a, b] \subset I$. *Then for every* $x \in [a, b]$, *the following inequality holds*:

$$|f(x) - L_{k-1}(x, f; \{x_0, h\})| \leqslant C(|x - x_0| + \rho_0)^{2k} \rho_0^{-2k} \bar{\omega}_\varphi^k(f, h, [a, b]).$$

In particular, for every $x \in [x_0, x_0 + (k-1)\rho_0]$,

$$|f(x) - L_{k-1}(x, f; \{x_0, h\}) \leqslant C\bar{\omega}_\varphi^k(f, h, [x_0, x_0 + (k-1)\rho_0]),$$

where $C = C(k)$.

The following lemma shows the connection between moduli of smoothness of different orders.

Lemma B [14, Lemma 18.4]. *Let* $k+1 \in N$, $r \in N$, $l = \overline{0, r-1}$, $(x, x + (k + r - l)\rho) \in I$, *and*

$$G_{k,l,r}(x, h) := \begin{cases} h^{2r-2l}\rho^{l-r}(w_{2l-r}(x, k+r-l, h))^{-1} & \text{if } l > r/2 \\ h^r\rho^{-r/2} |\ln(hw_1(x, k + r/2, h)\rho^{-1})| & \text{if } l = r/2 \\ h^r\rho^{-l} & \text{if } l < r/2. \end{cases}$$

If $f \in B^r$, *then*

$$\bar{\Delta}_\rho^{k+r-l}(f^{(l)}, x) \leqslant CG_{k,l,r}(x, h)\, \bar{\omega}_{\varphi,r}^k(f^{(r)}, h), \qquad h > 0, \tag{3}$$

where $C = C(r, k)$. *In particular, inequality* (1) *holds.*

Because of the importance of this lemma we shall quote its proof from [14].

Proof. First of all, it should be noted that here and in the proofs to follow we consider h to be small enough in comparison with r and k. This implies that for $y \in (x, x + (r-l)\rho)$ one can choose $\theta = \theta(y)$ so that $\rho = \rho(\theta h, y)$ and $0 < \theta \leqslant C$ for some constant C.

$$
\begin{aligned}
|\bar{\Delta}_\rho^k(f^{(r)}, y)| &= |\bar{\Delta}_{\rho(\theta h, y)}^k(f^{(r)}, y)| \\
&\leqslant C(1+y)^{-r/2}(1-y-k\rho(\theta h, y))^{-r/2}\,\bar{\omega}_{\varphi,r}^k(f^{(r)}, Ch) \\
&\leqslant C(1+y)^{-r/2}(1-y-k\rho)^{-r/2}\,\bar{\omega}_{\varphi,r}^k(f^{(r)}, h).
\end{aligned}
$$

Using the formulae for integral presentation of the usual differences we get

$$
\begin{aligned}
&|\bar{\Delta}_\rho^{k+r-l}(f^{(l)}, x)| \\
&\quad = \left| \int_0^\rho \cdots \int_0^\rho \bar{\Delta}_\rho^k(f^{(r)}, x+u_1+\cdots+u_{r-l})\, du_1 \cdots du_{r-l} \right| \\
&\quad \leqslant C\bar{\omega}_{\varphi,r}^k(f^{(r)}, h) \int_0^\rho \cdots \int_0^\rho (1+x+u_1+\cdots+u_{r-l})^{-r/2} \\
&\qquad \times (1-x-u_1-\cdots-u_{r-l}-k\rho)^{-r/2}\, du_1 \cdots du_{r-l} \\
&\quad \leqslant CG_{k,l,r}(x, h)\,\bar{\omega}_{\varphi,r}^k(f^{(r)}, h).
\end{aligned}
$$

Thus inequality (3) is proved. Inequality (1) is a consequence of (3) and of the estimate

$$
G_{k,l,r}(x, h) \leqslant Ch^{r-l}(w_l(x, k+r-l, h))^{-1}. \quad \blacksquare
$$

Remark. Another consequence of (3) is the estimate

$$
\bar{\omega}_\varphi^{k+r-l}(f^{(l)}, t) \leqslant Ct^{r-2l}\bar{\omega}_{\varphi,r}^k(f^{(r)}, t), \qquad t>0 \quad \text{and} \quad l<r/2.
$$

For $2l=r$ we have $\sup_x G_{k,l,r}(x, h) = +\infty$ which presents the main difficulty in this case. In fact, these will be the cases for $r=2$ and $r=4$ for monotone and convex approximations, respectively.

It turns out that the estimate (3) can be improved for some classes of functions. Namely, the following result is valid.

LEMMA 5. *Let $l \in N$, $f \in B^{2l}\bar{H}[k, \psi]$ with $\psi(t) = t^\beta \in \Phi^k$ (i.e., $0<\beta\leqslant k$) and such that $\beta<2l$. Then the following estimate holds*:

$$
|\bar{\Delta}_\rho^{k+l}(f^{(l)}, x)| \leqslant Ch^{2l}\rho^{-l}\psi(h), \qquad (x, x+(k+l)\rho) \subset (-1, 1)
$$

and $h>0$, where $C = C(k, l)(1/(2l-\beta))$.

Proof. The beginning of the proof is analogous to that of the previous lemma. First, using the formulae for integral presentation of the usual differences we get

$$\bar{\Delta}_\rho^{k+l}(f^{(l)}, x) = \bar{\Delta}_\rho^k \left(\int_0^\rho \cdots \int_0^\rho f^{(2l)}(\cdot + u_1 + u_2 + \cdots + u_l)\, du_1 \cdots du_l, x \right)$$

$$= \int_0^\rho \cdots \int_0^\rho \bar{\Delta}_\rho^k (f^{(2l)}, x + u_1 + u_2 + \cdots + u_l)\, du_1 \cdots du_l$$

$$=: \Theta(x).$$

Now if $[x, x+k\rho] \subset [-1+h^2, 1-h^2]$ and $x<0$ (for $x>0$ considerations are analogous), then

$$|\Theta(x)| \leqslant \int_0^\rho \cdots \int_0^\rho |\bar{\Delta}^k_{\rho(\theta h, y)}(f^{(2l)}, y)(1+y)^l (1-y-k\rho(\theta h, y))^l|$$

$$\times (1+y)^{-l} (1-y-k\rho(\theta h, y))^{-l}\, du_1 \cdots du_l,$$

where $y := x + u_1 + u_2 + \cdots + u_l$ and $\theta = \theta(y)$ is chosen so that $\rho(h, x) = \rho(\theta h, x + u_1 + u_2 + \cdots + u_l)$.

This yields

$$|\Theta(x)| \leqslant C \int_0^\rho \cdots \int_0^\rho \sup_{0<\tilde{h}<Ch}\ \sup_{y \in [-1+h^2, 1-h^2]}$$

$$\times |\bar{\Delta}^k_{\rho(\tilde{h}, y)}(f^{(2l)}, y)\, w_{2l}(y, k, \tilde{h})|\, (1+x)^{-l}\, du_1 \cdots du_l$$

$$\leqslant C \int_0^\rho \cdots \int_0^\rho \psi(Ch)(1+x)^{-l}\, du_1 \cdots du_l$$

$$\leqslant C\rho^l (1+x)^{-l} \psi(Ch)$$

$$\leqslant C h^{2l} \rho^{-l} \psi(h), \qquad C = C(k, l).$$

Now let us consider the case $-1 < x < -1 + h^2$ (if $[x, x+k\rho] \cap [1-h^2, 1] \neq \varnothing$, considerations are analogous).

The following Shevchuk's identity will be employed (see [14, identity (1.27)]):

Let $(N+1)$ given points $y_0, y_1, ..., y_N$ be such that $(M+1)$ of them coincide with $x_0, x_1, ..., x_M$, where $N \geqslant M \geqslant 2$. Then the following identity holds:

$$[x_0, x_1, ..., x_M; f] = \sum_{n=0}^{N-M} (y_{n+M} - y_n)[y_n, ..., y_{n+M}; f]$$

$$\times [x_0, ..., x_M; \Pi_{n,M}],$$

where $\Pi_{n,M}(x_s) := \prod_{j=1}^{M-1} (x_s - y_{n+j})_+$.

We fix $y := x + u_1 + u_2 + \cdots + u_l$, choose $m \geqslant 1$ so that $\rho/(2^m - 1) \leqslant y + 1 < \rho/(2^{m-1} - 1)$ and denote $v := \rho/(2^m - 1)$.

Let $m + k$ points z_i be defined by

$$z_i = y + (2^i - 1)v, \qquad i = \overline{0, m};$$
$$z_i = y + (i - m + 1)\rho, \qquad i = \overline{m+1, m+k-1}.$$

Now, $\bar{\Delta}_\rho^k(f^{(2l)}, y) = [y, y+\rho, ..., y+k\rho; f^{(2l)}]\, \rho^k k!$.

At the same time, for $k \geqslant 2$, choosing $M = k$ and $N = m + k - 1$, we have

$$[y, y+\rho, ..., y+k\rho; f^{(2l)}] = \sum_{n=0}^{m-1} (z_{n+k} - z_n)[z_n, ..., z_{n+k}; f^{(2l)}]$$
$$\times [y, y+\rho, ..., y+k\rho; \Pi_{n,k}],$$

where

$$\Pi_{n,k}(y + i\rho) := \prod_{j=1}^{k-1} (y + i\rho - z_{n+j})_+, \qquad i = \overline{0, k}. \tag{4}$$

Each term of this sum will be examined below.

First of all, let us note that $(z_{n+k} - z_n) \sim 2^n v$ for all $n = \overline{0, m-1}$. Now

$$\begin{aligned}
&|[z_n, ..., z_{n+k}; f^{(2l)}]| \\
&= \left| \frac{f^{(2l)}(z_{n+1}) - L_{k-1}(z_{n+1}; f^{(2l)}; z_n, z_{n+2}, z_{n+3}, ..., z_{n+k})}{(z_{n+1} - z_n)(z_{n+1} - z_{n+2})(z_{n+1} - z_{n+3}) \cdots (z_{n+1} - z_{n+k})} \right| \\
&\leqslant C(2^n v)^{-k}\, |f^{(2l)}(z_{n+1}) - \tilde{L}_{k-1}(z_{n+1}, f^{(2l)}; \{z_n, \tilde{h}\}) \\
&\quad - L_{k-1}(z_{n+1}; f^{(2l)} - \tilde{L}_{k-1}; z_n, z_{n+2}, z_{n+3}, ..., z_{n+k})|
\end{aligned}$$

where $\tilde{h}$ is chosen so that $z_n + (k-1)\,\rho(\tilde{h}, z_n) = z_{n+k}$, which implies that $\tilde{h} \sim \sqrt{2^n v}$.

Taking this into consideration and using Lemma A we have

$$\begin{aligned}
&|[z_n, ..., z_{n+k}; f^{(2l)}]| \\
&\quad \leqslant C(2^n v)^{-k}\, \bar{\omega}_\varphi^k(f^{(2l)}, \sqrt{2^n v}, [z_n, z_{n+k}]) \\
&\quad = C(2^n v)^{-k} \sup_{0 < \tilde{h} \leqslant \sqrt{2^n v}} \quad \sup_{z : [z, z + k\rho(z, \tilde{h})] \subset [z_n, z_{n+k}]} \\
&\qquad \times |\bar{\Delta}_{\rho(\tilde{h}, z)}^k (f^{(2l)}, z)| \\
&\quad \leqslant C(2^n v)^{-k-l} \sup_{\tilde{h}} \sup_{z} |w_{2l}(z, k, \tilde{h})\, \bar{\Delta}_{\rho(\tilde{h}, z)}^k (f^{(2l)}, z)| \\
&\quad \leqslant C(2^n v)^{-k-l}\, \psi(\sqrt{2^n v}), \qquad C = C(k, l).
\end{aligned}$$

Now $|[y, y+\rho, ..., y+k\rho; \Pi_{n,k}]|$ with $0 \leqslant n \leqslant m-1$ will be estimated. First, let us consider the case $n \leqslant m-k+1$. Let

$$p_{k-1}(z) := \prod_{j=1}^{k-1} (z - z_{n+j}), \qquad z \in [y, y+k\rho]$$

and

$$\tilde{p}_{k-1}(z) := \begin{cases} p_{k-1}(z) & \text{if } z \leqslant z_{n+k-1} \\ 0 & \text{otherwise.} \end{cases}$$

Then the equality

$$\Pi_{n,k}(z) := \prod_{j=1}^{k-1} (z - z_{n+j})_+ = p_{k-1}(z) - \tilde{p}_{k-1}(z)$$

gives

$$\begin{aligned} |[y, y+\rho, ..., y+k\rho; \Pi_{n,k}]| &= |[y, y+\rho, ..., y+k\rho; \tilde{p}_{k-1}]| \\ &= |\tilde{p}_{k-1}(y)| \, (k!\rho^k)^{-1} \leqslant C\rho^{-k}(2^n v)^{k-1}. \end{aligned}$$

Now if $m-k+1 < n \leqslant m-1$, then $2^n v \sim 2^m v \sim \rho$. This yields

$$\begin{aligned} |[y, y+\rho, ..., y+k\rho; \Pi_{n,k}]| &\leqslant C\rho^{-k} \sum_{i=0}^{k} \prod_{j=1}^{k-1} (y + i\rho - z_{n+j})_+ \\ &\leqslant C\rho^{-1} \leqslant C\rho^{-k}(2^n v)^{k-1}, \qquad C = C(k). \end{aligned}$$

Now putting all these estimates together we get the following:

$$|[y, y+\rho, ..., y+k\rho; f^{(2l)}]| \leqslant C(k, l)\, \rho^{-k} \sum_{n=0}^{m-1} (2^n v)^{-l} \psi(\sqrt{2^n v}).$$

Thus, using the inequality $\beta < 2l$, one has

$$\begin{aligned} |\bar{\Delta}_\rho^k(f^{(2l)}, y)| &\leqslant C(k, l) \sum_{n=0}^{\infty} (2^n(y+1))^{-l} \psi(\sqrt{2^n(y+1)}) \\ &\leqslant C(k, l) \sum_{n=0}^{\infty} (2^n(y+1))^{-l+\beta/2} \\ &\leqslant C(k, l)(y+1)^{-l+\beta/2} (1 - 2^{-l+\beta/2})^{-1} \\ &\leqslant C(k, l)(y+1)^{-l+\beta/2} \frac{1}{2l-\beta}. \end{aligned}$$

And now the desired estimate emerges as

$$|\Theta(x)| \leqslant C \int_0^\rho \cdots \int_0^\rho (x+u_1+\cdots+u_l+1)^{-l+\beta/2}\,du_1\cdots du_l$$

$$\leqslant C \begin{cases} |\bar{\Delta}^l_\rho((x+1)^{\beta/2}, x| & \text{if } \beta/2 \notin N \\ |\bar{\Delta}^l_\rho((x+1)^{\beta/2}\ln(x+1), x)| & \text{otherwise} \end{cases}$$

$$\leqslant Ch^\beta \qquad \text{where} \quad C = C(k, l)\frac{1}{2l-\beta}.$$

The last inequality is a consequence of Dzjadyk's [6, p. 160–161] understanding that if $\beta/2 \in N$ then $l \geqslant \beta/2+1$.

For $k=1$ considerations are simpler. The difference is that instead of (4) one should consider the identity

$$[y, y+\rho; f^{(2l)}] = \rho^{-1}\sum_{n=0}^{m-1}(z_{n+1}-z_n)[z_n, z_{n+1}; f^{(2l)}].$$

Thus the lemma is proved. ▮

In our proofs we shall deal with the first derivative of a function f in the monotone case and with the second one in the convex case. Obviously, the condition $f \in B^r$, $r \in N$, implies that $f \in C^\mu(-1, 1)$ for all $\mu \leqslant r$. However, it would be more convenient to have the continuity of the derivatives on the closed interval I.

The following lemma gives sufficient conditions for a function f to have continuous derivatives on $[-1, 1]$.

LEMMA 6. *Let* $k \in N$, $(r+1) \in N$, $\mu \in N$, $\psi \in \Phi^k$ *be such that* $\int_0^1 \psi(u)\,u^{-2\mu+r-1}\,du < +\infty$. *If for a function* f *and each* $n \geqslant k+r-1$ *there exists a polynomial* $p_n \in P_n$ *such that*

$$|f(x) - p_n(x)| \leqslant n^{-r}\psi(n^{-1}), \qquad x \in I,$$

then $f \in C^\mu[-1, 1]$.

Despite the fact that this lemma is probably known to the reader, its proof is adduced here since the author failed to find any references to it.

Proof. For any $n_0 \in N$ the series $\sum_{n=n_0}^M (p_{2^{n+1}}(x) - p_{2^n}(x))$ converges uniformly to $f(x) - p_{2^{n_0}}(x)$ as $M \to \infty$, and

$$|p_{2^{n+1}}(x) - p_{2^n}(x)| \leqslant 2^{1-nr}\psi(2^{-n}), \qquad x \in I.$$

Applying Markov's inequality one has

$$|p_{2^{n+1}}^{(\mu)}(x)-p_{2^n}^{(\mu)}(x)|\leqslant C2^{2n\mu-nr}\psi(2^{-n}), \qquad x\in I.$$

This implies

$$\sum_{n=n_0}^{\infty}|p_{2^{n+1}}^{(\mu)}(x)-p_{2^n}^{(\mu)}(x)|\leqslant C\sum_{n=n_0}^{\infty}\int_{2^{-n-1}}^{2^{-n}}u^{-2u+r-1}\psi(u)\,du$$

$$=C\int_0^{2^{-n_0}}u^{-2\mu+r-1}\psi(u)\,du<\infty.$$

Thus $f\in C^\mu[-1,1]$ and the proof of the lemma is complete. ∎

The following corollary is a consequence of Lemma 6 and Theorem A.

COROLLARY. *The following implications are valid*:

$$\begin{aligned}
&f\in B^3\bar{H}[k,\psi], &&\psi\in\Phi^k\Rightarrow f\in C^1[-1,1],\\
&f\in B^2\bar{H}[k,t^\beta], &&0<\beta\leqslant k\Rightarrow f\in C^1[-1,1],\\
&f\in B^5\bar{H}[k,\psi], &&\psi\in\Phi^k\Rightarrow f\in C^2[-1,1],\\
&f\in B^4\bar{H}[k,t^\beta], &&0<\beta\leqslant k\Rightarrow f\in C^2[-1,1].
\end{aligned}$$

LEMMA C (see [13], for example). *Let $p+1\in N$ and $q+1\in N$. The Dzjadyk-type kernel $D_{\zeta,n,\xi}(y,x)$ is a polynomial in x of degree $<(\xi+1)(n-1)$, and the following inequalities hold*:

$$\left|\frac{\partial^p}{\partial x^p}D_{\zeta,n,\xi}(y,x)\right|$$
$$\leqslant C\Delta^{\xi-p-1}(|x-y|+\Delta)^{-\xi}, \qquad C=C(p,\xi,\zeta);$$
$$\left|\frac{1}{p!}\int_{-1}^{1}(y-x)^q\frac{\partial^p}{\partial x^p}D_{\zeta,n,\xi}(y,x)\,dy-\delta_{p,q}\right|$$
$$\leqslant Cn^{-\min\{2\xi+1,\zeta+(1-(-1)^\zeta)/2\}}, \qquad C=C(p,q,\xi,\zeta),$$

where $\delta_{p,q}$ is the Kronecker symbol, and the integral in the last inequality is a polynomial of degree $\leqslant q-p$ (it is identically equal to zero if $q<p$).

Now let us note that the methods of proofs of Theorems 3–6 as well as of all auxiliary statements are the same. In connection with all this it would be inexpedient to give their proofs separately. In order to make this paper more readable and, on the other hand, not to lose in the fullness of exposition we shall give the complete statements of auxiliary propositions for all four cases, using the following abridgements. For the sake of convenience throughout the paper, in the wording $B^2\bar{H}[k, \psi]$ and $B^4\bar{H}[k, \psi]$ it will be implied that $\psi(t) = t^\beta, 0 < \beta \leqslant k, \ \beta < 2$ and $\psi(t) = t^\beta, \ 0 < \beta \leqslant k, \ \beta < 4$, respectively (however, most of the statements are true also for all functions $\psi \in \Phi^k$). We shall also use the notations [m_i], [m_ii], [c_i], and [c_ii] in order to emphasize cases designed for the proofs of Theorems 5, 3, 6, and 4, respectively. Also, we set variables Ξ and Λ to have the following definite values in these cases:

$$[\text{m\_i}]\ \Xi = 1, \Lambda = 2; \qquad [\text{m\_ii}]\ \Xi = 1, \Lambda = 3;$$

$$[\text{c\_i}]\ \Xi = 2, \Lambda = 4; \qquad [\text{c\_ii}]\ \Xi = 2, \Lambda = 5.$$

Thus, in order to follow the proof of Theorem 5, for example, it is enough to pay attention to the statements marked by [m_i], understanding that in this case $\Xi = 1$ and $\Lambda = 2$.

The following theorem is a generalization of the direct theorem (Theorem A) for $B^\Lambda\bar{H}[k, \psi]$ classes.

THEOREM 9. *Let a set $F \subset I$ and a function Q be such that $Q \in B^\Lambda\bar{H}[k, \psi]$ and $Q^{(\Xi)}(x) = 0$ for $x \in F$. Then the polynomial*

$$d_n(x, Q) := \int_{-1}^{1} (Q(y) - \Omega(y, Q))\, D_{\zeta, n, \xi}(y, x)\, dy + \Omega(x, Q)$$

approximates Q and its derivatives so that

$$|Q^{(p)}(x) - d_n^{(p)}(x, Q)|$$
$$\leqslant C_1 n^{-\Lambda} \Delta^{-p} \psi(n^{-1}) \left(\frac{\Delta}{\Delta + \operatorname{dist}(x, I \backslash F)}\right)^{\xi - 2k - 2\Lambda + \Xi - 1},$$
$$x \in I, \qquad p + 1 \in N, \qquad \text{and} \qquad 0 \leqslant p \leqslant \Xi,$$

where the polynomial $\Omega(x, Q)$ is defined by

$$[\text{m\_}]\quad \Omega(x, Q) := Q(-1) + \int_{-1}^{x} L_{k+\Lambda-2}(z, Q', \{-1, \sqrt{2(k+\Lambda-2)^{-1}}\})\, dz,$$

$$[\text{c\_}]\quad \Omega(x, Q) := Q(-1) + Q'(-1)(x+1) + \int_{-1}^{x} \int_{-1}^{t} L_{k+\Lambda-3} \times (z, Q'', \{-1, \sqrt{2(k+\Lambda-3)^{-1}}\})\, dz\, dt.$$

Proof. In order to avoid overloading of the text by unnecessary notations let us give the proof in the case [c_i]. The proofs for other cases are analogous.

Denote $g(x)=Q(x)-\Omega(x, Q)$. Then $g\in B^4\bar{H}[k,\psi]$ and applying Lemmas A, B, and 5 we get

$$|g(x)|=\left|\int_{-1}^{x}\int_{-1}^{t}\left(Q''(z)-L_{k+1}(z, Q'', \{-1, \sqrt{2(k+1)^{-1}}\})\right) dz\, dt\right.$$

$$\leqslant C\bar{\omega}_{k+2}(Q'', 1)\leqslant C\psi(1).$$

Thus

$$Q^{(p)}(x)-d_n^{(p)}(x, Q)=g^{(p)}(x)-\int_{-1}^{1} g(y)\frac{\partial^p}{\partial x^p} D_{\zeta,n,\xi}(y, x)\, dy.$$

Now let x be fixed and, for convenience, such that $x+(k+1)\Delta\leqslant 1$. Denote

$$l(y):=g(x)+g'(x)(y-x)+\int_x^y\int_x^t L_{k+1}(z, g'', \{x, n^{-1}\})\, dz\, dt$$

and note that $l^{(p)}(x)=g^{(p)}(x)$, $p=0, 1, 2$.

For $y\in[x, x+(k+1)\Delta]$ we have the estimate

$$|l(y)|\leqslant|l(y)-g(y)|+|g(y)|\leqslant C\psi(1)$$

$$+\int_x^y\int_x^t|L_{k+1}(z, g'', \{x, n^{-1}\})-g''(z)|\, dz\, dt$$

$$\leqslant C\psi(1)+C\bar{\omega}_\varphi^{k+2}(g'', n^{-1})\leqslant C\psi(1).$$

Therefore, applying Markov's inequality for all $j=\overline{0, k+3}$ we have

$$|l^{(j)}(y)|\leqslant C\Delta^{-j}\psi(1), \qquad y\in[x, x+(k+1)\Delta]$$

and, in particular, $|l^{(j)}(x)|\leqslant C\Delta^{-j}\psi(1)$.

We expand the polynomial $l(y)$ in Taylor series

$$l(y)=l(x)+\sum_{j=1}^{k+3}\frac{1}{j!}(y-x)^j\, l^{(j)}(x).$$

Thus

$$
\begin{aligned}
g^{(p)}(x) &- \int_{-1}^{1} g(y) \frac{\partial^p}{\partial x^p} D_{\zeta,n,\xi}(y,x)\,dy \\
&= \int_{-1}^{1} (l(y)-g(y)) \frac{\partial^p}{\partial x^p} D_{\zeta,n,\xi}(y,x)\,dy \\
&\quad + \sum_{j=0}^{k+3} \frac{1}{j!} l^{(j)}(x) \left(\delta_{j,p} p! - \int_{-1}^{1} (y-x)^j \frac{\partial^p}{\partial x^p} D_{\zeta,n,\xi}(y,x)\,dy \right) \\
&= A(x) + \sum_{j=0}^{k+3} \frac{1}{j!} B(x,j).
\end{aligned}
$$

Using Lemma C and the above estimate for $l^{(j)}(x)$ we have

$$
\begin{aligned}
\left| \sum_{j=0}^{k+3} \frac{1}{j!} B(x,j) \right| &\leqslant \sum_{j=0}^{k+3} C\Delta^{-j} \psi(1)\, n^{-\min\{2\xi+1, \zeta+(1-(-1)^{\zeta})/2\}} \\
&\leqslant C\Delta^{-k-3} \psi(1)\, n^{-\min\{2\xi+1, \zeta+(1-(-1)^{\zeta})/2\}} \\
&\leqslant C\Delta^{-k-3} \psi(n^{-1})\, n^{k-\min\{2\xi+1, \zeta+(1-(-1)^{\zeta})/2\}} \\
&\leqslant Cn^{-4} \Delta^{\xi-2k-p-7} \psi(n^{-1}).
\end{aligned}
$$

The last inequality is true if

$$
\min\{2\xi+1, \zeta+(1-(-1)^{\zeta})/2\} \geqslant 2\xi-k-4 \qquad \text{and} \qquad \xi \geqslant k+6.
$$

Now let us estimate $A(x)$, using Lemma C and the following estimate:

$$
\begin{aligned}
|l(y)-g(y)| &\leqslant \int_x^y \int_x^t |L_{k+1}(z, g'', \{x, n^{-1}\}) - g''(z)|\,dz\,dt \\
&\leqslant C\,|y-x|^2\, \bar{\omega}_{\varphi}^{k+2}(g'', n^{-1}) \left(\frac{|y-x|+\Delta}{\Delta} \right)^{2k+4} \\
&\leqslant C\,|y-x|^2 \left(\frac{|y-x|+\Delta}{\Delta} \right)^{2k+4} n^{-4} \Delta^{-2} \psi(n^{-1}), \qquad y \in I.
\end{aligned}
$$

Thus we have

$$
\begin{aligned}
|A(x)| &\leqslant C \int_{-1}^{1} |l(y)-g(y)|\, \Delta^{\xi-p-1} (|x-y|+\Delta)^{-\xi}\,dy \\
&\leqslant C \int_{-1}^{1} (|y-x|+\Delta)^{2k-\xi+6} \Delta^{\xi-p-2k-7} n^{-4} \psi(n^{-1})\,dy
\end{aligned}
$$

$$\leqslant C\,\Delta^{\xi-p-2k-7}n^{-4}\psi(n^{-1})\int_0^\infty (t+\Delta)^{2k-\xi+6}\,dt$$

$$\leqslant C\Delta^{-p}n^{-4}\psi(n^{-1}), \qquad \text{with} \quad \xi-2k-8\geqslant 0.$$

The estimate of the theorem is proved in the case $x\notin F$.

Now if $x\in F$, then $g''(x)=-L_{k+1}(x, Q'', \{-1, \sqrt{2(k+1)^{-1}}\})$, i.e., $g(x)$ is a polynomial of $(k+1)$th degree on F. Thus if $[x, x+(k+1)\Delta]\subset F$, then $l(y)=g(y)$ for $y\in F$, and therefore for these x

$$|A(x)|\leqslant C\int_{I\setminus F}(|y-x|+\Delta)^{2k-\xi+6}\,\Delta^{\xi-p-2k-7}n^{-4}\psi(n^{-1})\,dy$$

$$\leqslant Cn^{-4}\Delta^{-p}\psi(n^{-1})\left(\frac{\Delta}{\Delta+\operatorname{dist}(x, I\setminus F)}\right)^{\xi-2k-7}$$

The case when $x\in F$ and $(x+(k+1)\Delta)\notin F$ follows from the above, since in this case $\operatorname{dist}(x, I\setminus F)\sim\Delta$.

The proof is complete. ∎

Remark. In the proofs of Theorem 9 for the other cases it is sufficient to have the following inequalities: $\min\{2\xi+1, \zeta+(1-(-1)^{\zeta})/2\}\geqslant 2\xi-k-4$, $\xi-2k-10\geqslant 0$.

LEMMA 7 [m_] ([7], see also [13]). *Let E be a union of some intervals I_j. Then the polynomial*

$$\tilde{Q}_n(x, E):=\sum_{i\in\{i\mid I_{j_i}\in E\}}(T_{j_i,n}(x)-\tilde{T}_{j_i,n}(x))$$

of degree $\leqslant 6\chi(2n-1)+4n+1$ *satisfies the following inequalities*:

(1) $|\tilde{Q}_n(x, E)|\leqslant C_2, x\in I$;

(2) $\tilde{Q}_n'(x, E)\geqslant -C_3\Delta^{-1}, x\in E$;

(3) $\tilde{Q}_n'(x, E)\geqslant C_4\Delta^{-1}(\Delta/(\Delta+\operatorname{dist}(x, E)))^{12\chi-1}, x\in I\setminus E$.

LEMMA 7 [c_] (see [8]). *Let E be a union of some intervals I_j. Then the polynomial*

$$\tilde{Q}_n(x, E):=\sum_{i\in\{i\mid I_{j_i}\in E, I_{j_i+1}\in E\}}(x_{j_i-1}-x_{j_i})^{-1}\,(\sigma_{j_i,n}(x)-\tilde{\sigma}_{j_i,n}(x))$$

of degree $\leqslant 2(3\chi+1)(2n+1)$ *satisfies the inequalities*:

(1) $|\tilde{Q}_n(x, E)|\leqslant C_2,\ x\in I$;

(2) $\tilde{Q}_n''(x, E) \geqslant -C_3 \Delta^{-2}$, $x \in E$;

(3) $\tilde{Q}_n''(x, E) \geqslant C_4 \Delta^{-2}(\Delta/(\Delta + \operatorname{dist}(x, \tilde{E})))^{12\chi-2}$, $x \in I \backslash E$,

where $\tilde{E} := E \backslash \{I_{j_i} \mid I_{j_i \pm 1} \notin E\}$.

LEMMA 8 [m_] ([7], see also [13]). *Let* $0 \leqslant g'(x) \leqslant \Delta^{-1}$, $x \in I$, *then the polynomial* $R_n(x, g) := g(-1) + \sum_{j=1}^{n} (g(x_{j-1}) - g(x_j))\, T_{j,n}(x)$ *of degree* $\leqslant 6\chi(2n-1)+1$ *is nondecreasing on I, and the following inequality holds*:

$$|g(x) - R_n(x, g)| \leqslant C_5, \qquad x \in I.$$

LEMMA 8 [c_] ([8]). *Let* $0 \leqslant g''(x) \leqslant \Delta^{-2}$, $x \in I$, *then the polynomial*

$$\begin{aligned} R_n(x, g) := {} & g(x_{n-1}) + [x_n, x_{n-1}; g](x - x_{n-1}) \\ & + \sum_{j=1}^{n-1} [x_{j+1}, x_j, x_{j-1}; g](x_{j-1} - x_{j+1})\, \sigma_{j,n}(x) \end{aligned}$$

of degree $\leqslant 6\chi(2n-1)+2$ *is convex on I, and the following inequality holds*:

$$|g(x) - R_n(x, g)| \leqslant C_5, \qquad x \in I.$$

LEMMA 9. *Let a function* $g \in B^{\Lambda}\bar{H}[k, \psi]$ *and a set* ϑ_j, *which contains* $2k + 2\Lambda - 2\Xi - 1$ *neighboring intervals* I_j, *i.e.*, $\vartheta_j = I_j \cup I_{j+1} \cup \cdots \cup I_{j+2(k+\Lambda-\Xi-1)}$, *be given. If for every* $i = \overline{0, 2(k+\Lambda-\Xi-1)}$ *there exists a point* $\tilde{x}_i \in I_{j+i}$ *at which* $|g^{(\Xi)}(\tilde{x}_i)| \leqslant n^{-\Lambda}\psi(n^{-1})(\rho(n^{-1}, \tilde{x}_i))^{-\Xi}$, *then* $|g^{(\Xi)}(x)| \leqslant C_6 n^{-\Lambda}\psi(n^{-1})\, \Delta^{-\Xi}$ *for all* $x \in \vartheta_j$.

Proof. The identity

$$\begin{aligned} g^{(\Xi)}(x) = {} & (g^{(\Xi)}(x) - L_{k+\Lambda-\Xi-1}(x, g^{(\Xi)}, \{x_{j+2(k+\Lambda-\Xi-1)}, n^{-1}\})) \\ & - \tilde{L}_{k+\Lambda-\Xi-1}(x, g^{(\Xi)} - L_{k+\Lambda-\Xi-1}, \tilde{x}_0, \tilde{x}_2, \tilde{x}_4, ..., \tilde{x}_{2(k+\Lambda-\Xi-1)}) \\ & + \tilde{L}_{k+\Lambda-\Xi-1}(x, g^{(\Xi)}, \tilde{x}_0, \tilde{x}_2, \tilde{x}_4, ..., \tilde{x}_{2(k+\Lambda-\Xi-1)}), \end{aligned}$$

the inequality

$$\begin{aligned} & |g^{(\Xi)}(x) - L_{k+\Lambda-\Xi-1}(x, g^{(\Xi)}, \{x_{j+2(k+\Lambda-\Xi-1)}, n^{-1}\})| \\ & \quad \leqslant C\bar{\omega}_{\varphi}^{k+\Lambda-\Xi}(n^{-1}, g^{(\Xi)}, \vartheta_j) \leqslant Cn^{-\Lambda}\Delta^{-\Xi}\psi(n^{-1}), \qquad x \in \vartheta_j, \end{aligned}$$

which is a consequence of Lemmas A, B, and 5, and the estimate

$$|L_m(x, f; a_0, a_1, ..., a_m)| \leqslant \big(\max_{0 \leqslant i, j \leqslant m} |a_i - a_j|\big)^m \big(\min_{0 \leqslant i, j \leqslant m} |a_i - a_j|\big)^{-m} \max_{0 \leqslant i \leqslant m} |f(a_i)|$$

complete the proof of the lemma. ▮

6. Decomposition of Approximated Functions

Let a function f belong to $\Delta^{\Xi} \cap B^{\Lambda}\bar{H}[k, \psi]$.

Definition 1. The interval I_j will be called an *interval of type* I if, for all $x \in I_j$,

$$f^{(\Xi)}(x) \leqslant C_6(C_3 + C_4)\, n^{-\Lambda}\psi(n^{-1})\, \Delta^{-\Xi};$$

an *interval of type* II if it is not an interval of type I and, for all $x \in I_j$,

$$f^{(\Xi)}(x) \geqslant (C_3 + C_4)\, n^{-\Lambda}\psi(n^{-1})\, \Delta^{-\Xi}.$$

Let all other intervals be of *type* III.

We denote intervals of types I, II, and III by E_1, E_2, and E_3, respectively.

Remark. It follows from Lemma 9 that there cannot be more than $2(k + \Lambda - \Xi - 1)$ neighbouring intervals of type III; i.e., each set ϑ_j contains at least one interval of type I or II.

Now let the set [m_] $E_1 \cup E_3$ [c_] $E_1 \cup E_3 \cup \{I_j \in E_2 \mid I_{j \pm 1} \notin E_2\}$ be presented as a finite union of nonintersecting intervals. Let G_1 be the set containing all those intervals which include not less than $4k + 10$ intervals I_j:

$$G_1 = [x_{j_1}, x_{j_0}] \cup [x_{j_3}, x_{j_2}] \cup \cdots, \qquad 0 < j_\nu < j_{\nu+1} \leqslant n.$$

Let us denote $\bar{j}_\nu := j_\nu + \frac{1}{2}(1 + (-1)^\nu)$ and let $S_\nu(x) := 1$ if $|x_{j_\nu}| = 1$, and $S_\nu(x) := S(x, k+4; x_{j_\nu}, x_{\bar{j}_\nu})$ if $|x_{j_\nu}| \neq 1$ (see Section 4 for the definition of $S(x, l; a, b)$).

Definition 2. Let $g_1(x) := 0$ for $x \notin G_1$,

$$g_1(x) := f^{(\Xi)}(x)\, S_\nu(x) \qquad \text{for} \quad x \in [x_{j_\nu}, x_{\bar{j}_\nu}]$$

and $g_1(x) := f^{(\Xi)}(x)$ in all other cases.

Denote $g_2(x) := f^{(\Xi)}(x) - g_1(x)$ and

$$[\mathrm{m}_-] \quad f_1(x) := f(-1) + \int_{-1}^{x} g_1(y)\, dy,$$

$$f_2(x) := \int_{-1}^{x} g_2(y)\, dy;$$

$$[\mathrm{c}_-] \quad f_1(x) := f(-1) + f'(-1)(x+1) + \int_{-1}^{x} \int_{-1}^{t} g_1(y)\, dy\, dt,$$

$$f_2(x) := \int_{-1}^{x} \int_{-1}^{t} g_2(y)\, dy\, dt.$$

Obviously, the following correlations hold:

$$f_1(x) + f_2(x) = f(x),$$

$$g_1(x) \geqslant 0 \text{ and } g_2(x) \geqslant 0 \qquad \text{for all} \quad x \in I.$$

LEMMA 10. *The following inequality holds*:

$$g_1(x) \leqslant C_7 n^{-\Lambda} \psi(n^{-1})\, \Delta^{-\Xi}, \qquad x \in I.$$

Proof. Analogously to the proof of Lemma 9, one can show the validity of the estimate $f^{(\Xi)}(x) \leqslant Cn^{-\Lambda}\psi(n^{-1})\, \Delta^{-\Xi}$, $x \in G_1$. Together with $0 \leqslant S_\nu(x) \leqslant 1$, this proves the lemma. ▮

LEMMA 11. *The function f_2 belongs to $B^{\Lambda}\bar{H}[k, C_8\psi]$.*

Lemma 11 is a consequence of the following lemmas.

LEMMA 12 [m_]. *Let the interval $[a, b] \subset [-1 + n^2, 1 - n^{-2}]$ be such that $|a - b| \sim \sqrt{1 - a^2}/n$, where n is a fixed natural number which is sufficiently large. And let a given function $g \in B^r\bar{H}[k, \psi]$, $r \geqslant 1$, be such that*

$$|g'(x)| \leqslant n^{-r}\psi(n^{-1})(b-a)^{-1}, \qquad x \in [a, b].$$

Define the function G so that

$$G'(x) := g'(x)\, S(x, l; a, b), \qquad -1 \leqslant x \leqslant 1$$

and $G(-1) = g(-1)$, where $l \geqslant k + r$.

Then $G \in B^r\bar{H}[k, C\psi]$ with C independent of n.

LEMMA 12 [c_]. *Let the interval $[a, b] \subset [-1 + n^2, 1 - n^{-2}]$ be such that $|a - b| \sim \sqrt{1 - a^2}/n$, where n is a fixed natural number which* *is sufficiently large. And let a given function* $g \in B^r\bar{H}[k, \psi]$, $r \geqslant 2$, *be such that*

$$|g''(x)| \leqslant n^{-r}\psi(n^{-1})(b-a)^{-2}, \qquad x \in [a, b].$$

Define the function G *so that* $G''(x) := g''(x)\, S(x, l; a, b)$, $-1 \leqslant x \leqslant 1$, $G(-1) = g(-1)$, *and* $G'(-1) = g'(-1)$, *where* $l \geqslant k + r$. *Then* $G \in B^r\bar{H}[k, C\psi]$ *with* C *independent of* n.

We shall prove Lemma 12 [c_] for $[a, b] \subset [-1+n^2, 0]$ (for $b \geqslant 0$ considerations are similar). The case [m_] is analogous with the only difference being that instead of the second derivatives one should deal with the first ones.

Proof of Lemma 12 [c_]. We shall use the fact that the interval $[a, b]$ is separated from the endpoints of the interval $[-1, 1]$. Also, it is enough to consider the behavior of G "near" the interval $[a, b]$, as outside of $[a, b]$ G either is a linear function or it coincides with g.

Namely, it is sufficient to prove that

$$\sup_{0<h\leqslant t}\ \sup_{[x,x+k\rho]\cap[a,b]\neq\varnothing} |(1+x)^{r/2}\, \bar{\Delta}^k_\rho(G^{(r)}, x)| \leqslant C\psi(t),$$

where t is such that $t < (10kn)^{-1}$.

Now let $0 < h \leqslant t$ be fixed and note that if $[x, x+k\rho] \cap [a, b] \neq \varnothing$ then $x \in [a - 3k\sqrt{1+a}\,h, b]$ and the following holds:

$$(1+x) \sim (1+a) \qquad \text{and} \qquad \rho \sim \sqrt{1+x}\,h \sim \sqrt{1+a}\,h. \tag{5}$$

For convenience denote $S(x) := S(x, l, a, b)$ and note that

$$|S^{(p)}(x)| \leqslant C(b-a)^{-p}, \qquad p = \overline{0, l},\ x \in I. \tag{6}$$

(In fact, $S^{(p)}(x) = 0$ for $x \notin [a, b]$ and $p \geqslant 1$).

We shall use the Marchaud inequality for the usual moduli of smoothness (see, for example, [5, 6, 14])

$$\omega_j(t, g; [a, b]) \leqslant Ct^j \left(\int_t^{b-a} u^{-j-1}\omega_k(u, g; [a, b])\, du + (b-a)^{-j}\, \|g\|_{[a,b]} \right), \qquad j = \overline{1, k-1},$$

and the Besov inequality [2] which is given by

$$\|g^{(j)}\|_{[a,b]} \leqslant C((b-a)^{r-j}\, \omega_k(b-a, g^{(r)}; [a, b]) + (b-a)^{-j}\, \|g\|_{[a,b]}), \qquad j = \overline{0, r}.$$

Now using (6) and also the identities

$$G^{(r)}=(g''S)^{(r-2)}=\sum_{i=0}^{r-2}\binom{r-2}{i}g^{(i+2)}S^{(r-i-2)}$$

and

$$\bar{\Delta}_h^k(g_1g_2, x_0)=\sum_{j=0}^{k}\binom{k}{j}\bar{\Delta}_h^j(g_1, x_0)\,\bar{\Delta}_h^{k-j}(g_2, x_0+jh),$$

we have for $x_0\in[a-3k\sqrt{1+a}\,h, b]$

$$|\bar{\Delta}_\rho^k(G^{(r)}, x_0)|\leqslant C\sum_{i=0}^{r-2}\sum_{j=0}^{k}\binom{r-2}{i}\binom{k}{j}\rho^{k-j}$$
$$\times(b-a)^{i+j+2-r-k}\,|\bar{\Delta}_\rho^j(g^{(2+i)}, x_0)|.$$

Now (5) and the Besov inequality yield, for $0\leqslant i\leqslant r-2$,

$$\begin{aligned}\|g^{(i+2)}\|_{[a,b]}&\leqslant C((b-a)^{r-i-2}\,\omega_k(b-a, g^{(r)}; [a,b])\\&\quad+(b-a)^{-i}\,\|g''\|_{[a,b]})\\&\leqslant C((b-a)^{r-i-2}\,(1+a)^{-r/2}\,\psi((b-a)(1+a)^{-1/2})\\&\quad+(b-a)^{-i-2}\,n^{-r}\psi(n^{-1}))\\&\leqslant C(b-a)^{r-i-2}\,(1+a)^{-r/2}\,\psi(n^{-1}).\end{aligned}$$

Using the last estimate, (5), and the Marchaud inequality, we have for $j<k+r-i-2$:

$$\begin{aligned}&\bar{\omega}_\varphi^j(t, g^{(i+2)}; [a,b])\\&\quad\sim\omega_j(\sqrt{1+a}\,t, g^{(i+2)}; [a,b])\\&\quad\leqslant C(1+a)^{j/2}\,t^j\left(\int_{\sqrt{1+a}\,t}^{b-a}u^{-j-1}\omega_{k+r-i-2}(u, g^{(i+2)}; [a,b])\,du\right.\\&\quad\left.+(b-a)^{-j}\,\|g^{(i+2)}\|_{[a,b]}\right)\\&\quad\leqslant C(1+a)^{j/2}\,t^j\left(\int_{\sqrt{1+a}\,t}^{b-a}u^{r-i-j-3}(1+a)^{-r/2}\,\psi(u(1+a)^{-1/2})\,du\right.\\&\quad\left.+(b-a)^{r-i-j-2}\,(1+a)^{-r/2}\,\psi(n^{-1})\right)\end{aligned}$$

$$\leqslant C(1+a)^{(j-r)/2}\, t^{j-k}\psi(t)$$
$$\times\left((1+a)^{-k/2}\int_{\sqrt{1+a}\,t}^{b-a} u^{k+r-i-j-3}\,du + n^{-k}(b-a)^{r-i-j-2}\right)$$
$$\leqslant C(1+a)^{(j-r)/2}\, t^{j-k}\psi(t)\, n^{-k}(b-a)^{r-i-j-2}. \tag{7}$$

Note that $k+r-i-j=2$ only if $i=r-2$ and $j=k$, and thus $\bar{\omega}_\varphi^k(t, g^{(r)}; [a,b]) \leqslant C(1+a)^{-r/2}\psi(t)$; i.e., (7) is true in this case also.

Now putting all these estimates together we have the following inequalities for any x_0, such that $[x_0, x_0+k\rho(h, x_0)] \cap [a,b] \neq \varnothing$:

$$|(1+x_0)^{r/2}\,\bar{\Delta}_\rho^k(G^{(r)}, x_0)|$$
$$\leqslant C\sum_{i=0}^{r-2}\sum_{j=0}^{k}\binom{r-2}{i}\binom{k}{j}\rho^{k-j}(b-a)^{-k}(1+a)^{j/2}\, n^{-k} t^{j-k}\psi(t)$$
$$\leqslant C\sum_{i=0}^{r-2}\sum_{j=0}^{k}\binom{r-2}{i}\binom{k}{j}(b-a)^{-k}(1+a)^{k/2}\, n^{-k}\psi(t)$$
$$\leqslant C\psi(t).$$

Thus the lemma is proved. ∎

Denote $\mathscr{E} := \{I_j \mid I_j \in E_2, I_j \notin G_1\}$ (clearly, $\mathscr{E} = E_2$ in the case [m$_-$]) and $G_2 := \{x \mid \operatorname{dist}(x, \mathscr{E}) \leqslant 3^{2k+7}\Delta\}$.

It follows from Definition 2 that $g_2(x) = 0$ for $x \in I \backslash G_2$. Note that for $n_1 \geqslant n$ the following inequality holds:

$$\frac{\rho(n_1^{-1}, x)}{\operatorname{dist}(x, G_2) + \rho(n_1^{-1}, x)} \leqslant C_9\,\frac{\Delta}{\operatorname{dist}(x, \mathscr{E}) + \Delta}.$$

Now we choose ξ, ζ, and χ so that all the conditions in the proofs above are valid. For example, $\xi = 24k$, $\zeta = 48k$, and $\chi = k$ will do.

The following lemma is a consequence of Theorem 9 and Lemma 11.

LEMMA 13. *For any integer $n_1 \geqslant n$ the polynomial $d_{n_1}(x, f_2)$ has the following properties*:

$$|f_2(x) - d_{n_1}(x, f_2)| \leqslant C_{10} n^{-A}\psi(n^{-1}),$$
$$d_{n_1}^{(\Xi)}(x, f_2) \geqslant -C_{11} n_1^{-A}\psi(n_1^{-1})(\rho(n_1^{-1}, x))^{-\Xi}$$
$$\times\left(\frac{\Delta}{\Delta + \operatorname{dist}(x, \mathscr{E})}\right)^{12k}, \qquad x \in I\backslash\mathscr{E}$$

and

$$d_{n_1}^{(\Xi)}(x, f_2) \geq (C_3 + C_4)\, n^{-\Lambda}\, \psi(n^{-1})\, \Delta^{-\Xi}$$
$$- C_{11} n_1^{-\Lambda} \psi(n_1^{-1})(\rho(n_1^{-1}, x))^{-\Xi}, \qquad x \in \mathscr{E},$$

where $C_{10} = C_1 C_8$ *and* $C_{11} = C_1 C_8 C_9^{12k}$.

7. Proofs of Theorems 3–6

Let $n_1 \in N$, $n_1 \geq n$. Denote

$$\pi_{n_1}(x) := n^{-\Lambda} \psi(n^{-1})\, \tilde{Q}_n(x, \mathscr{E}) + d_{n_1}(x, f_2) + R_n(x, f_1).$$

Then $\pi_{n_1}(x)$ is a polynomial of degree $< 50kn_1$.

It follows from Lemmas 7, 8, 10, and 13 that

$$|f(x) - \pi_{n_1}(x)| \leq (C_2 + C_{10} + C_5 C_7)\, n^{-\Lambda}\, \psi(n^{-1}), \qquad x \in I,$$
$$\pi_{n_1}^{(\Xi)}(x) \geq (C_4 C_{12} n^{-\Lambda} \psi(n^{-1})\, \Delta^{-\Xi}$$
$$- C_{11} n_1^{-\Lambda} \psi(n_1^{-1})(\rho(n_1^{-1}, x))^{-\Xi})$$
$$\times \left(\frac{\Delta}{\Delta + \operatorname{dist}(x, \mathscr{E})}\right)^{12k}, \qquad x \in I \backslash \mathscr{E}$$

and

$$\pi_{n_1}^{(\Xi)}(x) \geq C_4 n^{-\Lambda} \psi(n^{-1})\, \Delta^{-\Xi}$$
$$- C_{11} n_1^{-\Lambda} \psi(n_1^{-1})(\rho(n_1^{-1}, x))^{-\Xi}, \qquad x \in \mathscr{E},$$

where $C_{12} = (3^{8k+23})^{12k}$.

Now let us choose n_1 so that $n_1 = C_{13} n$, where $C_{13} := \{[4C_{11}/C_4 C_{12}] + 2\} \in N$. Then the following inequalities hold

[m_i] $\pi'_{n_1}(x) > 0$, $x \in I \backslash (I_1 \cup I_n)$,
$\pi'_{n_1}(x) > -C_{11} \psi(n^{-1})$, $x \in I_1 \cup I_n$;

[m_ii] $\pi'_{n_1}(x) > 0$, $x \in I$;

[c_i] $\pi''_{n_1}(x) > 0$, $x \in I \backslash (I_1 \cup I_n)$,
$\pi''_{n_1}(x) > -C_{11} \psi(n^{-1})$, $x \in I_1 \cup I_n$;

[c_ii] $\pi''_{n_1}(x) > 0$, $x \in I$.

Thus Theorems 3 and 4 are proved for $n \geq C_{13}$.

In order to obtain analogous results for Theorems 5 and 6, the following lemmas will be useful.

LEMMA 14 [m_]. *For the algebraic polynomial of degree* $< 5n$,

$$M_n(x) := \int_{-1}^{x} \left(\frac{\sin(n/2 \arccos t)}{n \sin(1/2 \arccos t)} \right)^{10} dt,$$

the following inequalities hold:

$$\begin{aligned} &M_n'(x) \geqslant 0, && x \in I, \\ &0 \leqslant M_n(x) \leqslant 10^5 n^{-2}, && x \in I, \\ &M_n'(x) \geqslant 2^{-10}, && x \in I_1. \end{aligned}$$

LEMMA 14 [c_]. *For the algebraic polynomial of degree* $< 5n$,

$$\mathcal{M}_n(x) := \int_{-1}^{x} \int_{-1}^{y} \left(\frac{\sin(n/2 \arccos t)}{n \sin(1/2 \arccos t)} \right)^{10} dt\, dy,$$

the following inequalities hold:

$$\begin{aligned} \mathcal{M}_n''(x) &\geqslant 0, && x \in I, \\ O \leqslant \mathcal{M}_n(x) &\leqslant 2 \times 10^4 n^{-4}, && x \in I, \\ \mathcal{M}_n''(x) &\geqslant 2^{-10}, && x \in I_1. \end{aligned}$$

Proof. Lemma 14 [c_] is Lemma 8 from [8]. Lemma 14 [m_] can be verified by direct computations with the use of inequalities $2t/\pi \leqslant \sin t \leqslant t$, $0 \leqslant t \leqslant \pi/2$, or, applying Markov's inequality, it can be immediately derived from Lemma 14 [c_]. ▮

Now the polynomial

$$[\mathrm{m}\_] \qquad \bar{\pi}_n(x) := \pi_{n_1}(x) + 2^{10} C_{11} \psi(n^{-1})(M_n(x) - M_n(-x)),$$

$$[\mathrm{c}\_] \qquad \bar{\pi}_n(x) := \pi_{n_1}(x) + 2^{10} C_{11} \psi(n^{-1})(\mathcal{M}_n(x) + \mathcal{M}_n(-x)),$$

of degree $< 50kn_1$ satisfies Theorem 5 in the monotone case and Theorem 6 in the convex one.

Thus Theorems 5 and 6 are proved for $n \geqslant C_{13}$.

For the other n, the theorems are consequences of the cases $n = k + 1$ for Theorem 5, $n = k + 2$ for Theorem 3, $n = k + 3$ for Theorem 6, and $n = k + 4$ for Theorem 4, for which it is sufficient to choose

$$\pi_n(x) := \Omega(x, f) + 5 \max\{C_0(2, k), C_0(3, k), C_0(4, k), C_0(5, k)\}\psi\left(\sqrt{\frac{2}{k}}\right) x^{\Xi}.$$

The proofs of Theorems 3–6 are now complete.

ACKNOWLEDGMENTS

The author is indebted to I. A. Shevchuk for useful discussions and valuable suggestions, and also to the referees who helped to make this paper more readable.

REFERENCES

1. N. K. BARI AND S. B. STECHKIN, Best approximations and differential properties of two conjugate functions, *Trud. Moskov. Mat. Obsch.* **5** (1956), 483–522.
2. O. V. BESOV, Continuation of functions with conservation of their differential and difference properties in L_p, *Dokl. Akad. Nauk USSR* **150**, No. 3 (1963), 963–966.
3. R. A. DEVORE, Monotone approximation by polynomials, *SIAM J. Math. Anal.* **8**, No. 5 (1977), 906–921.
4. R. A. DEVORE, Monotone approximation by splines, *SIAM J. Math. Anal.* **8**, No. 5 (1977), 891–905.
5. Z. DITZIAN AND V. TOTIK, "Moduli of Smoothness," Springer-Verlag, Berlin, 1987.
6. V. K. DZJADYK, "Introduction to the Theory of Uniform Approximation of Functions by Polynomials," Izdat. "Nauka", Moscow, 1977.
7. G. A. DZUBENKO, V. V. LISTOPAD, AND I. A. SHEVCHUK, Uniform estimates of monotone polynomial approximation, *Ukr. Mat. Zurnal* **45**, No. 1 (1993), 38–43.
8. K. A. KOPOTUN, Uniform estimates of convex approximation of functions by polynomials, *Mat. Zametki* **51**, No. 3 (1992), 35–46: Engl. transl., *Math. Notes* **51**, No. 3 (1992), 245–254.
9. K. A. KOPOTUN, Pointwise and uniform estimates for convex approximation of functions by algebraic polynomials, *Constr. Approx.* **10**, No. 2 (1994), 153–178.
10. K. A. KOPOTUN AND V. V. LISTOPAD, Remarks on monotone and convex approximation by algebraic polynomials, *Ukr. Mat. Zurnal* **46**, No. 9 (1994).
11. D. LEVIATAN, Monotone and comonotone approximation revisited, *J. Approx. Theory* **53** (1988), 1–16.
12. D. LEVIATAN, Pointwise estimates for convex polynomial approximation, *Proc. Amer. Math. Soc.* **98**, No. 3 (1986), 471–474.
13. I. A. SHEVCHUK, On coapproximation of monotone functions, *Dokl. Akad. Nauk USSR* **308**, No. 3 (1989), 537–541.

14. I. A. SHEVCHUK, "Approximation by Polynomials and Traces of the Functions Continuous on an Interval," Naukova dumka, Kiev, 1992.
15. A. S. SHVEDOV, Orders of coapproximation of functions by algebraic polynomials, *Mat. Zametki*, **29**, No. 1 (1981), 117–130; Engl. transl. *Math. Notes* **30** (1981), 63–70.
16. S. B. STECHKIN, On the order of best approximations of continuous functions, *Izv. Akad. Nauk SSSR, Ser. Mat.* **15**, No. 3 (1951), 219–242.